# Exercise 1

Use the *series solution method* to solve the Volterra integral equations of the first kind:

$$e^x - 1 - x = \int_0^x (x - t + 1)u(t)\,dt$$

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### Solution

We seek a series solution for $u$:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

Substitute this and the Taylor series expansion of $e^x$,

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots,$$

into the integral equation.

$$\begin{aligned}
\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots\right) - 1 - x &= \int_0^x (x - t + 1)(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots)\,dt \\
\frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots &= a_0 \int_0^x (x - t + 1)\,dt + a_1 \int_0^x (x - t + 1)t\,dt \\
&\quad + a_2 \int_0^x (x - t + 1)t^2\,dt + a_3 \int_0^x (x - t + 1)t^3\,dt \\
&\quad + a_4 \int_0^x (x - t + 1)t^4\,dt + \cdots \\
&= a_0\left(\frac{x^2}{2} + x\right) + a_1\left(\frac{x^3}{6} + \frac{x^2}{2}\right) + a_2\left(\frac{x^4}{12} + \frac{x^3}{3}\right) \\
&\quad + a_3\left(\frac{x^5}{20} + \frac{x^4}{4}\right) + a_4\left(\frac{x^6}{30} + \frac{x^5}{5}\right) + \cdots \\
&= a_0 x + \frac{1}{2}(a_0 + a_1)x^2 + \frac{1}{6}(a_1 + 2a_2)x^3 \\
&\quad + \frac{1}{12}(a_2 + 3a_3)x^4 + \frac{1}{20}(a_3 + 4a_4)x^5 + \cdots
\end{aligned}$$

Match the coefficients of the powers of $x$ to obtain a system of equations for $a_i$.

$$\begin{aligned}
a_0 &= 0 \\
\frac{1}{2}(a_0 + a_1) &= \frac{1}{2} \quad \rightarrow \quad a_1 = 1 \\
\frac{1}{6}(a_1 + 2a_2) &= \frac{1}{6} \quad \rightarrow \quad a_2 = 0 \\
\frac{1}{12}(a_2 + 3a_3) &= \frac{1}{24} \quad \rightarrow \quad a_3 = \frac{1}{6} \\
\frac{1}{20}(a_3 + 4a_4) &= \frac{1}{120} \quad \rightarrow \quad a_4 = 0
\end{aligned}$$

Notice that the general recurrence relation is

$$\frac{1}{(n+2)(n+1)}[a_n + (n+1)a_{n+1}] = \frac{1}{(n+2)!}.$$

Solve it for $a_{n+1}$

$$a_{n+1} = \frac{1}{n+1}\left(\frac{1}{n!} - a_n\right)$$

and use this result to determine the rest of the $a_i$.

$$\begin{aligned} a_5 &= \frac{1}{120} \\ a_6 &= 0 \\ a_7 &= \frac{1}{5040} \\ a_8 &= 0 \\ &\vdots \end{aligned}$$

The solution is then

$$u(x) = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \cdots,$$

which is the known Taylor series of $\sinh x$. Therefore,

$$u(x) = \sinh x.$$